KGÜ 4

A 14

X - stetige Zufallsvariable (Z.V.) auf $(\Omega, \mathcal{F}, P)$

$f_X(x)$ - Dichtefkt, $f_X(x) = \begin{cases} \frac{1}{3}(5-4x), & x \in (0,1) \\ 0 & \text{, sonst} \end{cases}$

$Y = aX, \; a > 0$

Dichtefkt von Y?

Verteilungsfkt von Y?

Nach Voraussetzung: X stetige Z.v. mit Werten in $\mathbb{R}$ and $f_X(x)$ Dichtefkt.

Definiere: $g : (0,1) \mapsto (0,a), \; x \mapsto g(x) := a \cdot x, \; a > 0$

g ist surjektiv: $\forall\, y \in (0,a)\ \exists$ ein $x \in (0,1)$, sodaß $y = ax$ $\left(x = \frac{y}{a}\right)$, $y \in (0,a) \overset{a>0}{\Rightarrow} \frac{y}{a} = x \in (0,1)$.

g ist injektiv: $\forall\, x_1, x_2 \in (0,1)$ mit $x_1 \neq x_2$ gilt sofort

$$g(x_1) = ax_1 \neq ax_2 = g(2).$$

Somit ist g bijektiv. Umkehrfunktion von g:

$$g^{-1} : (0,a) \mapsto (0,1), \quad y \mapsto g^{-1}(y) = \frac{y}{a}.$$

g und g' stetig diff'bare Fkt,

$$(g^{-1})'(y) = \left(\frac{y}{a}\right)' = \frac{1}{a} \cdot (y') = \frac{1}{a} \quad \forall\, y \in (0,a)$$

Nach dem Dichtentransformationssatz:

für $Y = g(X)$:

$$f_Y(y) = f_X(g^{-1}(y)) \cdot \left| \frac{d\, g^{-1}(y)}{dy} \right| .$$

$$f_X(g^{-1}(y)) = f_X\left(\frac{y}{a}\right) = \frac{1}{3}\left(5 - 4\cdot\frac{y}{a}\right) 1\!\!1_{(0,1)}\left(\frac{y}{a}\right) =$$

$$= \frac{1}{3}\left(5 - 4\frac{y}{a}\right) 1\!\!1_{(0,a)}(y). \quad \frac{y}{a} \in (0,a)$$

es folgt $f_Y(y) = \frac{1}{3a}\left(5 - 4\frac{y}{a}\right) 1\!\!1_{(0,a)}(y).$

$0 < \frac{y}{a} < 1 \mid \cdot a, \; a > 0$

$0 < y < a$

$\Rightarrow 1\!\!1_{(0,a)}(y)$

$$F_Y(z) = \int_{-\infty}^{z} f_Y(y)\,dy = \int_{-\infty}^{z} \frac{1}{3a}\left(5 - 4\cdot\frac{y}{a}\right) 1\!\!1_{(0,a)}(y)\,dy$$

1. $z \le 0$: $F_Y(z) = \int_{-\infty}^{z} 0\,dy = 0$

2. $z \in (0,a)$: $F_Y(y) = \int_0^z \frac{1}{3a}\left(5 - 4\frac{y}{a}\right)dy =$

$$= \frac{1}{3a}\left[5y - \frac{4}{a}\cdot\frac{1}{2}y^2\right]_0^z = \frac{5}{3a}z - \frac{2}{3a^2}z^2$$

3. $z \ge a$:

$$F_Y(z) = \int_0^a \frac{1}{3a}\left(5 - 4\cdot\frac{y}{a}\right)dy =$$

$$= \frac{1}{3a}\left[5y - \frac{4}{a}\cdot\frac{1}{2}y^2\right]_0^a = \frac{5}{3} - \frac{2}{3} = 1$$

also:
$$F_Y(z) = \begin{cases} 0 & , z \le 0 \\ \frac{5z}{3a} - \frac{2}{3a^2}z^2 & , z \in (0,a) \\ 1 & , z \ge a. \end{cases}$$

c) $E(X) = \sum_{i\in\{-1,0,1\}} i \cdot P(X=i) = -1\cdot \underbrace{P(X=-1)}_{1/4} + \underbrace{P(X=1)}_{1/4}\cdot 1 + 0 \cdot \underbrace{P(X=0)}_{0} =$

$= -\frac{1}{4} + \frac{1}{4} = 0$

$E(Y) = \sum_{j\in\{1,2,3\}} j \cdot P(Y=j) = 1\cdot P(Y=1) + 2\cdot P(Y=2) + 3\cdot P(Y=3) =$

$= \frac{7}{20} + 2\cdot\frac{1}{2} + 3\cdot\frac{3}{20} = \frac{9}{5}$

$Var(X) = E(X^2) - (E(X))^2 =$

$= \sum_{i\in\{-1,0,1\}} i^2 \cdot P(X=i) - 0^2 =$

$= (-1)^2 \cdot P(X=-1) + 0^2 \cdot P(X=0) + 1^2 \cdot P(X=1) =$

$= \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$

$Var(Y) = \cancel{E(Y)^2}\ E(Y^2) - (E(Y))^2 =$

$= \sum_{j\in\{1,2,3\}} j^2 P(Y=j) - \left(\frac{9}{5}\right)^2 =$

$= \frac{7}{20} + 4\cdot\frac{1}{2} + 9\cdot\frac{3}{20} - \left(\frac{9}{5}\right)^2 =$

$= \frac{37}{10} - \left(\frac{9}{5}\right)^2 = \frac{23}{50}$

A15

$X: -1, 0, 1$

$Y: 1, 2, 3$

$p_{ij} = P(X=i, Y=j)$, $i \in \{-1, 0, 1\}$, $j \in \{1, 2, 3\}$

$p_{ij} = P(X=i, Y=j)$

$i \backslash j$	1	2	3	
−1	v) 1/20	iv) 1/5	0	1/4
0	1/5	1/5	iii) 1/10	1/2 ii)
1	1/10	1/10	viii) 1/20	1/4
	vi) 7/20	1/2	3/20 vii)	1

a)

i) Σ aller Wahrscheinlichkeiten muss 1 ergeben

ii) $1 - \frac{1}{4} - \frac{1}{4} = \frac{2}{4} = \frac{1}{2}$

iii) $\frac{1}{2} - \frac{1}{5} - \frac{1}{5} = \frac{1}{10}$

iv) $\frac{1}{2} - \frac{1}{5} - \frac{1}{10} = \frac{1}{5}$

v) $\frac{1}{4} - \frac{1}{5} - 0 = \frac{1}{20}$

vi) $\frac{1}{20} + \frac{1}{5} + \frac{1}{10} = \frac{7}{20}$

vii) $1 - \frac{7}{20} - \frac{1}{2} = \frac{3}{20}$

viii) $\frac{3}{20} - \frac{1}{10} - 0 = \frac{1}{20}$

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b) X, Y s.u. falls $P(X=i, Y=j) = P(X=i) \cdot P(Y=j) \ \forall i, j \in \ldots$

↯ : $P(X=-1, Y=3) = 0 \neq \frac{3}{80} = \frac{1}{4} \cdot \frac{3}{20} = P(X=-1) \cdot P(Y=3)$

$\Rightarrow X, Y$ nicht s.u.

b) $E(e^X) =$

$$= \int_{-\infty}^{\infty} u\, f_e(u)\, du =$$

$$= \int_0^{\infty} \frac{1}{\sqrt{2\pi}} \cancel{u} \cdot \frac{1}{u} \cdot e^{-\frac{1}{2}\log^2(u)}\, du = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} \cdot \underbrace{e^z dz}_{du}$$

$z = \log(u)$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{z^2}{2} + z}\, dz =$$

$$= e^{1/2} \cdot \underbrace{\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(z-1)^2}\, dz}_{= 1 \text{ (Dichte der } \mathcal{N}(1,1)\text{-Verteilung)}} = e^{1/2}$$

$z = \log(u)$ $\quad \log u = z$

$e^z = u$ $\quad u = e^z$

$e^z dz = du$, also $du = e^z dz$ ("$z = \log(\infty)$; $e^{\infty} = \infty$ also $z = \infty$")

↑ Grenze: $u = \infty$, $z = \log(u) = \cancel{\log}\ z = \infty$

↓ Grenze $u = 0$, $z = \log(u) \Rightarrow z = -\infty$

("$z = \log(0)$; $e^z = 0$ also $z = -\infty$ da $\lim_{z \to -\infty} e^z = 0$")

A16

$X \sim \mathcal{N}(0,1)$

$\varphi(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$, $x \in \mathbb{R}$

a) $F_{e^X}(y) = P(e^X \leq y) =$

$= P(\log(e^X) \leq \log(y)) =$

$= P(X \leq \log(y)) =$

$= \int_{-\infty}^{\log(y)} \varphi(z) dz =$

$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz$ du

Substitution: $\cancel{z} = \log(u)$

$u = e^z$

$\Leftrightarrow du = e^z dz$

$\Leftrightarrow dz = \frac{du}{e^z} \Leftrightarrow dz = \frac{du}{u}$

↓ Grenze: $z = -\infty$

$z = \log(u)$

$-\infty = \log(u)$

$\lim_{b \to -\infty} e^{-\cancel{b}} = \lim_{b \to -\infty} \frac{1}{e^b} = \underline{\underline{0.}}$

F $\int_0^y \dots = F(y)$

↑ Grenze: $z = \log(y)$

$z = \log(u) = \log(y) \Leftarrow u = \underline{y}$

$\int_a^b = F(b) - F(a)$

$F(y)$

$\Rightarrow \int_{-\infty}^{\log(y)} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz = \int_0^y \frac{1}{\sqrt{2\pi}} \cdot e^{-\frac{\log^2(u)}{2}} \cdot \frac{1}{u} du = F(y)$

und damit hat e^X die Dichte

$f_{e^X}(y) = \frac{1}{\sqrt{2\pi} \cdot y} e^{-\frac{\log^2(y)}{2}} \underline{\mathbb{1}_{(0,\infty)}(y)}.$

A17

paarweise s.u. X, Y, Z auf $(\Omega, \mathcal{F}, P)$.

$E(X)=2$ $\quad$ $E(Y)=1$ $\quad$ $E(Z)=11$

$E(X^2)=5$ $\quad$ $E(Y^2)=3$

Sei $A := 5X-7Y$.

a) $E(A) \overset{\text{Linearität}}{=} E(5X-7Y) = 5E(X) - 7E(Y) = 5\cdot 2 - 7\cdot 1 = 3$

b) $Var(A) = Var(5X-7Y) \underset{X,Y \text{ s.u.}}{=} 5^2 Var(X) + (-7)^2 Var(Y) =$

$= 25 Var(X) + 49 Var(Y) =$

$= 25(E(X^2) - (E(X))^2) + 49(E(Y^2) - (E(Y))^2) =$

$= 25(5-2^2) + 49(3-1^2) =$

$= 25 \cdot 1 + 49 \cdot 2 = 123$

* Verschiebungssatz:
$Var(X) = E(X^2) - (E(X))^2$ *

c) $E(A \cdot X) =$ A, X NICHT s.u.

$= E((5X-7Y)\cdot X) =$

$= E(5X^2 - 7XY) =$

$= 5E(X^2) - 7E(X\cdot Y) =$ X, Y s.u.

$= 5E(X^2) - 7E(X)\cdot E(Y) =$

$= 5\cdot 5 - 7\cdot 2\cdot 1 =$

$= 25 - 14 = \underline{\underline{11}}$ □

d) $E(A\cdot Z) \underset{A, Z \text{ s.u.}}{=} E(A)\cdot E(Z) = 3\cdot 11 = 33$

$\frac{7}{20} + \frac{20}{20} + \frac{9}{20} = \frac{36}{20} = \cancel{\frac{36}{20}}$

$\frac{\cancel{6}\,3}{\cancel{4}\,2}$ $\frac{3 \cdot \frac{1}{2}}{}$ $1{,}5$

$1 \cdot \frac{1}{2}$ $1 + \frac{1}{2}$ $1 + \frac{1}{2} = 1{,}5$

$1 \cdot \frac{1}{2} \to \frac{1}{2}$

$\frac{36}{20} = \frac{20}{20} + \frac{16}{20} =$

$= 1 + \frac{16}{20} =$

$= 1 + \frac{4}{5}$

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