KGÜ 6

A22

X, Y s.u. ZV. auf $(\Omega, \mathcal{F}, P)$

$X \sim U[0,6]$

$Y \sim Exp(\frac{1}{2})$.

a) $E(X+Y) = ?$ b) $Var(X-Y)$, c) ..... d) ... e) ...

nach Voraussetzung:

- $X \sim U[0,6] \Rightarrow E(X) = \frac{0+6}{2} = 3$

  $Var(X) = \frac{(6-0)^2}{12} = \frac{36}{12} = 3$

- $Y \sim Exp(\frac{1}{2}) \Rightarrow E(Y) = \frac{1}{\frac{1}{2}} = 2$

  $Var(Y) = \frac{1}{(\frac{1}{2})^2} = \frac{1}{\frac{1}{4}} = 4$

zugehörige Verteilungsfkt:

$$F_X(z) = P(X \leq z) = \begin{cases} 0 & , z \leq 0 \\ \frac{z-0}{6-0} = \frac{z}{6} & , 0 < z < 6 \\ 1 & , z \geq 6 \end{cases} \quad , z \in \mathbb{R}.$$

$$F_Y(z) = P(Y \leq z) = \begin{cases} 0 , & z \leq 0 \\ 1 - e^{-\frac{1}{2}z} & , z > 0 \end{cases}$$

X, Y stochastich unabhängig

a) $E(X+Y) = E(X) + E(Y) = 3 + 2 = 5$

b) X, Y s.u. $\Rightarrow Var(X-Y) = Var(X) + Var(-Y) =$

$= Var(X) + (-1)^2 Var(Y) = Var(X) + Var(Y) =$

$= 3 + 4 = 7.$

(2)

c) $E(XY) \underset{\underset{X,Y\ n.u}{\uparrow}}{=} E(X)\cdot E(Y) = 3\cdot 2 = 6$

d) $P(X \geq 6, Y \leq 2) \overset{X,Y\ n.u}{=}$

$= P(X \geq 6)\cdot P(Y \leq 2) =$

$= (1 - P(X \leq 6))\cdot P(Y \leq 2) =$

$= (1 - F_X(6))\cdot F_Y(2) =$

$= (1-1)\cdot F_Y(2) =$

$= 0.$

e) $P(X \leq 4, Y \leq \ln(4)) \overset{X,Y\ n.u.}{=}$

$= P(X \leq 4)\cdot P(Y \leq \ln(4)) =$

$= F_X(4)\cdot F_Y(\ln(4)) =$

$= \frac{4}{6}\left(1 - e^{-\frac{1}{2}\ln(4)}\right) =$

$-\frac{1}{2}\ln(4) = \ln(4^{-\frac{1}{2}}) = \ln\left(\frac{1}{4^{+\frac{1}{2}}}\right) =$
$= \ln\left(\frac{1}{\sqrt{4}}\right) = \ln\left(\sqrt{\frac{1}{4}}\right).$

$= \frac{2}{3}\left(1 - e^{\ln(\sqrt{\frac{1}{4}})}\right) = \frac{2}{3}\left(1 - \frac{1}{2}\right) = \frac{2}{3}\cdot\frac{1}{2} = \frac{1}{3}$

A23

X, Y Zv. mit Dichtefkt. $f_c(x,y) = \begin{cases} cy^2(2-x-y) & , 0<x<1, \; 0<y<1 \\ 0 & , \text{sonst} \end{cases}$

a) f_c nur für $c=4$ Dichtefkt.

1\. Fall $0<x<1$ und $0<y<1$:

$$f_c(x,y) = c \underbrace{y^2}_{\geq 0} \underbrace{(2-x-y)}_{>0, \text{ da } 0<x<1 \text{ und } 0<y<1} \geq 0 \iff \underline{c \geq 0}$$

2\. im anderen Fall gilt immer $f_c(x,y) \geq 0$.

Noch:

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_c(x,y)\,dx\,dy \overset{!}{=} 1$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_c(x,y)\,dx\,dy = \int_0^1\int_0^1 cy^2(2-x-y)\,dx\,dy = c\int_0^1\int_0^1 (2y^2-2xy^2-y^3)\,dy$$

$$= c\int_0^1 \left[2xy^2 - \frac{1}{2}x^2y^2 - xy^3\right]_{x=0}^{x=1} dy =$$

$$= c\int_0^1 \left(2y^2 - \frac{1}{2}y^2 - y^3\right) dy =$$

$$= c\int_0^1 \left(\frac{3}{2}y^2 - y^3\right) dy = c\left[\frac{1}{2}y^3 - \frac{1}{4}y^4\right]_{y=0}^{y=1} = c\left(\frac{1}{2} - \frac{1}{4}\right) = \frac{c}{4} = 1$$

$$\iff \underline{\underline{c = 4}}$$

$(y^3)' = 3y^2 \qquad \left(\frac{1}{2}y^3\right)' = \frac{1}{2}\cdot 3y^2 = \frac{3}{2}y^2$

~~b) E(X)=?, E(Y)=?~~

~~für $x \in (0,1)$:~~

b) f_X, f_Y von X, Y. (4)

- $x \in (0,1)$:

$$f_X(x) = \int_{-\infty}^{\infty} f_c(x,y)\,dy = \int_0^1 4y^2(2-x-y)\,dy = 4\left[\frac{2}{3}y^3 - \frac{1}{3}xy^3 - \frac{1}{4}y^4\right]_{y=0}^{y=1}$$

$$= 4\left(\frac{2}{3} - \frac{1}{3}x - \frac{1}{4}\right) = \frac{5}{3} - \frac{4}{3}\cdot x$$

- $x \notin (0,1)$: $f_X(x) = \int_{-\infty}^{\infty} 0\,dy = 0.$

- $y \in (0,1)$:

$$f_Y(y) = \int_{-\infty}^{\infty} f_c(x,y)\,dx = \int_0^1 4y^2(2-x-y)\,dx = 4y^2\left[2x - \frac{1}{2}x^2 - xy\right]_{x=0}^{x=1}$$

$$= 4y^2\left(2 - \frac{1}{2} - y\right) = 6y^2 - 4y^3$$

- $y \notin (0,1)$: $f_Y(y) = 0.$

c) $E(X), E(Y)$:

$$E(X) = \int_{-\infty}^{\infty} x f_X(x)\,dx = \int_0^1 x\left(\frac{5}{3} - \frac{4}{3}x\right)dx = \left[\frac{5}{6}x^2 - \frac{4}{9}x^2\right]_0^1 = \frac{5}{6} - \frac{4}{9} = \frac{7}{18}$$

$$E(Y) = \int_{-\infty}^{\infty} y f_Y(y)\,dy = \int_0^1 (6y^3 - 4y^4)\,dy = \left[\frac{3}{2}y^4 - \frac{4}{5}y^5\right]_0^1 = \frac{3}{2} - \frac{4}{5} = \frac{7}{10}$$

d) $\underset{23/324}{Var}(X) = E(X^2) - (E(X))^2 =$

$$= \int_{-\infty}^{\infty} x^2 \cdot f_X(x)\,dx - \left(\frac{7}{18}\right)^2 =$$

$$= \int_0^1 x^2\left(\frac{5}{3} - \frac{4}{3}x\right)dx - \left(\frac{7}{18}\right)^2 =$$

$$= \left[\frac{5}{9}x^3 - \frac{1}{3}x^4\right]_0^1 - \left(\frac{7}{18}\right)^2 = \frac{5}{9} - \frac{1}{3} - \left(\frac{7}{18}\right)^2 = \frac{2}{9} - \left(\frac{7}{18}\right)^2 =$$

$$= \frac{23}{324}.$$

$\underset{13/300}{Var}(Y) = E(Y^2) - (E(Y))^2 = \int_{-\infty}^{\infty} y^2 f_Y(y)\,dy =$

$$= \int_0^1 (6y^4 - 4y^5)\,dy - \left(\frac{7}{10}\right)^2 = \left[\frac{6}{5}y^5 - \frac{2}{3}y^6\right]_0^1 - \left(\frac{7}{10}\right)^2 = \frac{6}{5} - \frac{2}{3} - \left(\frac{7}{10}\right)^2 =$$

$$= 13/300$$

e) $Cov(X,Y) = E(XY) - E(X) \cdot E(Y)$

$$E(XY) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy\, f_e(x,y)\, dx\, dy =$$

$$= \int_0^1 \int_0^1 4xy^3(2-x-y)\, dx\, dy =$$

$$= \int_0^1 \left[2xy^4 - x^2y^4 - \frac{4}{5}xy^5\right]_{y=0}^{y=1} dx =$$

$$= \int_0^1 (2x - x^2 - \frac{4}{5}x)\, dx = \left[\frac{3}{5}x^2 - \frac{1}{3}x^3\right]_0^1 = \frac{3}{5} - \frac{1}{3} = \frac{4}{15}$$

$$Cov(X,Y) = \frac{4}{15} - \frac{7}{18} \cdot \frac{7}{10} = -\frac{1}{180}$$

f) $$Cor(X,Y) = \frac{Cov(X,Y)}{\sqrt{Var(X)} \cdot \sqrt{Var(Y)}} = \frac{-\frac{1}{180}}{\sqrt{\frac{23}{324}} \cdot \sqrt{\frac{13}{300}}} \approx -0.1$$

g) X, Y s.u?

Nein, da $Cov(X,Y) \neq 0$ also X, Y nicht s.u. (siehe e))

A24 ⑥

$Y \sim Bin(10, \frac{1}{2})$

$Z \sim Poi(2)$, $Cov(Y,Z) = 1$.

3-dim. Zufallsvektor $X = (X_1, X_2, X_3)'$

$$X_1 := 4Y$$
$$X_2 := 2Y - 3Z$$
$$X_3 := -Z$$

1) Erwartungsvektor μ_X =?

2) Kovarianzmatrix $Cov(X)$ von X =?

$Y \sim Bin(10, \frac{1}{2}) \Rightarrow E(Y) = 10 \cdot \frac{1}{2} = 5$

$Var(Y) = 10 \cdot \frac{1}{2}(1 - \frac{1}{2}) = \frac{5}{2}$

$Z \sim Poi(2) \Rightarrow E(Z) = 2$

~~E(Z)=2~~

$Var(Z) = 2$.

$$\mu_X = E(X) = \begin{pmatrix} E(X_1) \\ E(X_2) \\ E(X_3) \end{pmatrix} = \begin{pmatrix} 4E(Y) \\ 2E(Y) - 3E(Z) \\ -E(Z) \end{pmatrix} = \begin{pmatrix} 20 \\ 4 \\ -2 \end{pmatrix}$$

$$Cov(X) = \begin{pmatrix} Var(X_1) & Cov(X_1, X_2) & Cov(X_1, X_3) \\ Cov(X_2, X_1) & Var(X_2) & Cov(X_2, X_3) \\ Cov(X_3, X_1) & Cov(X_3, X_2) & Var(X_3) \end{pmatrix}.$$

! Y und Z NICHT s.u., da $Cov(Y,Z) = 1 \neq 0$.

- $Var(X_1) = Var(4Y) = 4^2 Var(Y) = 16 \cdot \frac{5}{2} = 40$
- $Var(X_2) = Var(2Y - 3Z) = \overset{(2^2)}{4} Var(Y) + \overset{((-3)^2)}{9} Var(Z) + 2 \cdot 2 \cdot (-3) Cov(Y,Z) =$
  $= 10 + 18 - 12 \cdot 1 = 28 - 12 = 16$
- $Var(X_3) = Var(-Z) = (-1)^2 Var(Z) = 2$.

$$Cov(X_1,X_2) = Cov(4Y, 2Y-3Z) = Cov(4Y,2Y) - Cov(4Y,-3Z) =$$
$$= 8\,Var(Y) - 12\,Cov(Y,Z) =$$
$$= 20 - 12\cdot 1 = 8$$

$$Cov(X_1,X_3) = Cov(4Y,-Z) = -4\,Cov(Y,Z) = -4\cdot 1 = -4$$

$$Cov(X_2,X_3) = Cov(2Y-3Z, -Z) = Cov(2Y,-Z) + Cov(-3Z,-Z) =$$
$$= -2\,Cov(Y,Z) + 3\,Var(Z) =$$
$$= -2+6 = 4.$$

Kovarianz Symmetrie!

$$Cov(X) = \begin{pmatrix} 40 & 8 & -4 \\ 8 & 16 & 4 \\ -4 & 4 & 2 \end{pmatrix}.$$

<u>A25</u>

$X = (X_1,X_2)'$ 2-dim. normalverteilt Zv. mit Erw.vektor μ_X,

$\mu_X = 0 \in \mathbb{R}^2$ und Kovm. $\Sigma = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

$\rightarrow$

Kovm. von $\overset{Zv.}{Y} = (Y_1,Y_2)'$, $Y_1 = X_1 - X_2$
$Y_2 = X_1 + X_2$

Sind Y_1, Y_2 s.u?

$\rightarrow$

$Cov(X_1,X_2) = Cov(X_2,X_1) = 0$ (Voraussetzung), da X_1, X_2 Komponenten eines 2-dim. verteilten Zv sind, sind X_1,X_2 <u>s.u</u>

! "Aus $Cov(X_1,X_2)=0$ folgt X_1,X_2 s.u" <u>NUR</u> wenn X_1,X_2 normalverteilt sind.
Für beliebige Verteilungen ist diese Aussage im Allgemeinen falsch.

Aufgrund Unabhängigkeit.

$$Var(Y_1) = Var(X_1 - X_2) = Var(X_1) + (-1)^2\,Var(X_2) = 1+1 = 2.$$
$$Var(Y_2) = Var(X_1+X_2) = Var(X_1) + Var(X_2) = 1+1 = 2$$
$$Cov(Y_1,Y_2) = Cov(X_1-X_2, X_1+X_2) = Cov(X_1,X_1) + Cov(X_1,X_2) - Cov(X_2,X_1) - Cov(X_2,X_2) = 1+0-0-1 = 0.$$

Somit:

$\Sigma_Y = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$.

Die Summe (bzw. Differenz) von unabhängigen normalverteilten Zufallsvar wieder normalverteilt ist, sind Y_1, Y_2 jeweils auch normalverteilt.

Zusammen mit $Cov(Y_1, Y_2) = 0$ folgt Y_1, Y_2 s.u.